

Equations of Motion for Nonaxisymmetric Vibrations of Prolate Spheroidal Shells

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PREFACE

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13. ABSTRACT (Maximum 200 words) This report documents the derivation of the equations for nonaxisymmetric motion of prolate spheroidal shells of constant thickness. The equations include the effect of distributed mechanical surface forces and moments. These equations were derived from the Lagrangian of the system. The thin shell theory used in this derivation includes three displacements and two changes of curvature. Thus, the effects of membrane, bending, shear deformations, and rotatory inertias are included in this theory. The resulting five coupled partial differential equations are self-adjoint and positive definite, ensuring real and positive eigenvalues and real eigenfunctions. The frequency-wavenumber spectrum has five branches (three acoustic and two optical) representing flexural, longitudinal, torsional, and thickness-shear modes. The acoustic branches have the correct group and phase velocities, especially in the high wavenumber, short wavelength limits.				
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EQUATIONS OF MOTION FOR NONAXISYMMETRIC VIBRATIONS OF PROLATE SPHEROIDAL SHELLS

1. INTRODUCTION

Prolate spheroidal shells (PSS) are shells of revolution that can span a wide range of shapes, from spherical to needle-like. They are essentially ellipsoids of revolution about their major axis and may have a constant thickness or a variable thickness defined by two confocal ellipses. The ellipse is characterized by an interfocal length d and eccentricity $1/a$, where a is the radial coordinate of the shell's mid-surface in the prolate spheroidal coordinate system. In the limits $d \rightarrow 0$, $a \rightarrow \infty$, and $(da/2) \rightarrow R$, the ellipse approaches a circle whose radius is R and the shell then becomes spherical.

The theory of thin elastic shells is based on the approximation that the shell thickness h is much smaller than the typical dimension of the shell R , i.e., $h/R \ll 1$. Typically, the ratio of h/R is taken to be less than 0.05. Unlike thin elastic plates, the in-plane and out-of-plane displacements of thin elastic shells are coupled. Furthermore, unlike elastic plates, elastic shells can deform in an extensional mode (membrane) or in a coupled extensional/flexure mode (membrane and bending).

In the extensional theory of thin shells of revolution, there are three independent displacements: w is the out-of-plane displacement, u is an in-plane displacement (extension) in the plane of the generator surface, and v is a displacement perpendicular to the generator surface, i.e., torsional. The displacements u and w are coupled in longitudinal and flexural deformation, while v is uncoupled and is purely torsional deformation. Therefore, the strain energy density is exclusively dependent on the extensional strains of the shell's mid-surface and is linearly dependent on the thickness h/R . Thus, for the extensional theory of shells, there are three branches in the frequency-wavenumber spectrum corresponding to the three displacements u , v , and w . The lowest branch is an acoustic branch representing flexural vibrations of the shell where $|w/u| > 1$. This acoustic branch has a constant limit in the frequency as the wavenumber $k \rightarrow \infty$. This means that the group and phase velocities for this branch approach zero as $k \rightarrow \infty$, rather than the sound speed of the shell material. Furthermore, the flexural resonance frequencies are not distinct for $k \rightarrow \infty$, since the frequency spacing approaches zero as the mode number k becomes very large. The theory predicts the low-ordered mode resonance frequencies accurately, but becomes inaccurate after the first few modes. The second acoustic branch represents a decoupled (pure) torsional displacement v , which has group and phase velocities correctly approaching the shear sound speed of the shell material, i.e., $c_s^2 = G / \rho_s$ as $k \rightarrow \infty$, where G is the shear modulus,

and ρ_s is the density. The third branch is an optical branch that represents $|w/u| < 1$, i.e., primarily in-plane deformation of the shell, representing longitudinal vibrations. As $k \rightarrow \infty$, this branch has phase and group velocities equal to the sound speed in plate materials (i.e., $c_p^2 = E/(\rho_s(1 - \nu^2))$, where E is the Young's modulus and ν is the Poisson's ratio), which are close to the exact values.

In the coupled extensional-bending theory of thin shells, $h/R \ll 1$ is again assumed. However, the change of the curvatures of the shell's mid-surface β_1 and β_2 are now included in the strain energy density of the shell. These changes of curvature (bending) are expressed in terms of the derivatives of u , v , and w , so that the strain energy density is again dependent on only these three displacements. The strain energy has membrane components that are linearly dependent on the thickness (h/R), and additional bending components have cubic dependence on the thickness, i.e., (h^3/R^3) . Thus, there are still three branches in the frequency-wavenumber spectrum corresponding to the three independent variables u , v , and w . The lower acoustic branch representing the flexural vibration is dependent on terms of (h/R) and (h^3/R^3) . However, the group and phase velocities become infinite as $k \rightarrow \infty$, i.e., $\omega \propto k^2$, similar to the Bernoulli-Euler bending theory of elastic plates. Thus, the flexural vibration is accurate in the low frequency range but is still inaccurate in the high-frequency range. The second acoustic branch is the extensional torsional mode, which is not influenced by the addition of bending. The optical branch representing longitudinal vibrations is slightly influenced by the bending terms; and thus, the group and phase velocities are good approximations to the exact values in the limit of $k \rightarrow \infty$.

To improve the accuracy of the flexural vibration branch for high frequencies, the thin shell theory is improved by the addition of shear deformations and rotatory inertias. This means that the change of curvatures β_1 and β_2 are no longer derivatives of u , v , and w , but are independent variables. The new thin-shell theory now has five independent variables, (u , v , w , β_1 , and β_2); the frequency-wavenumber spectrum has five branches: two acoustic branches (one torsional and one flexural) and three optical branches (one longitudinal and two thickness shear modes). The flexural branch now has the correct group and phase velocities.

This report first surveys the literature covering prolate spheroids and PSS for free and forced vibrations of unloaded shells, fluid-filled shells, free and forced vibrations of submerged shells, and acoustic scattering from submerged shells. It then presents the development of the five coupled partial differential equations for thin elastic shells, including extensional, bending, shear deformation and rotatory inertias. These equations are developed from the Lagrangian of the strain and kinetic energy densities of the shell, using Hamilton's variational principle. This approach leads to an accurate, self-consistent coupled set of five partial differential equations of the correct expansion in h/R valid to $O(h^3/R^3)$. This approach also guarantees a self-adjoint positive-definite system of equations that leads to real and positive eigenvalues and real eigenfunctions.

2. LITERATURE REVIEW

2.1 FREE AND FORCED VIBRATIONS OF UNLOADED PROLATE SPHEROIDAL SHELLS (PSS)

DiMaggio and Silbiger¹ were the first to examine the free extensional axisymmetric torsional vibrations of a PSS of variable thickness. The equations of motion were based on the Lagrangian of the shell using Hamilton's variational principle. The exact solution was found in terms of the angular prolate spheroidal wave function $S_{on}(\eta)$. Silbiger and DiMaggio² developed the equations for axisymmetric extensional flexural vibration of PSS of a variable thickness. Using the Rayleigh-Ritz method, they obtained the resonance frequencies and mode shapes, but the solution was limited to low eccentricity of the shell (≈ 0.7). Shiraishi and DiMaggio³ obtained the solution by using perturbation techniques. The perturbation parameter was the eccentricity ($1/a$) and their perturbation series was extended to $O(1/a^4)$. It should be noted that the zero order term is the spherical shell resonances and mode shapes. Again these frequencies were valid for low eccentricity (≈ 0.7). Nemergut and Brand⁴ developed the extensional flexural vibration of a constant-thickness shell using a numerical integration scheme. DiMaggio and Rand⁵ obtained the solution of variable-thickness flexural extensional vibration by finite difference techniques. Rand⁶ obtained the exact solution for torsional vibration of solid prolate spheroids and prolate spheroidal shells using the exact theory of elasticity.

Burroughs and Magrab⁷ were the first to include bending and shear deformation in the shell equations using a variational approach in terms of shear and moment resultants. They developed five coupled equations for nonaxisymmetric motion for a shell with constant thickness. However, they obtained solutions for only the axisymmetric vibration using Galerkin's method, which means that there are three branches for the axisymmetric case. They showed that for axisymmetric vibrations, the resonance frequencies for the flexural modes increase with (h/R) , while the longitudinal modes are not influenced by the addition of bending or shear deformations. Of course, there is a new branch representing thickness-shear mode that was shown to be inversely proportional to (h/R) . Yahner and Burroughs⁸ obtained the axisymmetric vibration of an open shell with a variable thickness. They exhibited the influence of the opening of the shell on the three axisymmetric modes, flexural, longitudinal, and thickness-shear. Chen and Ginsberg⁹ revisited the problem by focusing on the loci of the eigenvalues of three PSS for the axisymmetric vibration, but they included only extensional and bending terms and used the Galerkin method in their solution.

2.2 VIBRATION OF FLUID-LOADED PSS

The first attempt at examining the fluid loading effects on the vibration of submerged PSS was made by Hayek and DiMaggio.^{10,11} The complex natural frequencies of submerged PSS using only extensional strain energy were found for shells of variable thickness using perturbation techniques on the strain energy density and the surface acoustic pressure. They showed that the two pairs of real frequencies ($\pm\omega_{1,2}$) of the flexural and longitudinal branches of in-vacuo shells have increased to seven frequencies: three pairs of complex frequencies ($\pm\omega_{1,2,3} + i\xi_{1,2,3}$) and one purely imaginary frequency ($i\xi_4$). The two pairs with small imaginary components ($\pm\omega_{1,2} + i\xi_{1,2}$) of the submerged shells correspond to the two real pairs of the unloaded shells. The real parts ($\pm\omega_{1,2}$) decreased from the in-vacuo shells due to the additional fluid mass loading of the acoustic medium, and the imaginary parts represent the energy lost through acoustic radiation (acoustic resistance) from the shell. The remaining pair ($\pm\omega_3 + i\xi_3$) has a large imaginary part, caused by high acoustic resistance. The purely imaginary root implies pure damping. Rand and DiMaggio¹² examined the resonance frequencies of extensional vibration of fluid-filled PSS. The presence of a fluid inside the shell increases the number of resonance frequencies because of the three-dimensionality of the cavity resonances of the interior of the PSS. Yen and DiMaggio¹³ obtained the solution for the forced vibration of submerged PSS due to axisymmetric time-harmonic surface forces. They used numerical integration techniques after transforming the infinite exterior acoustic medium domain to a finite domain. Bedrosian and DiMaggio¹⁴ obtained the transient solution of an axisymmetric extensional vibration of a submerged PSS undergoing sudden application of a uniform surface source at $t = 0$.

Berger¹⁵ was the first to include bending effects in the problem of forced vibration of a submerged PSS of constant thickness. He used a finite difference approach for the shell equations and mapped the infinite region of the exterior acoustic medium into a finite region. Berger presented the response of the shell attributed to a normal transient loading. Lee and DiMaggio¹⁶ obtained the response of a fluid-filled PSS as a model of a human head. They included bending theory in their model and obtained the response of the PSS due to harmonic surface sources. Ross and Johns¹⁷ obtained the solution for the axisymmetric vibration of a free and submerged (both sides) hemi-PSS and conducted experiments to support their analytic predictions. Prikhod'ko¹⁸ obtained the solution for a slender submerged PSS, but the solution is valid for sufficiently slender shells in order to neglect the fluid loading on the shell ends. Pauwelussen¹⁹ used a finite element method (FEM) to solve the problem of axisymmetric response of submerged PSS due to a shock load. Chen and Ginsberg²⁰ obtained the acoustic radiation for PSS including bending effects due to axisymmetric mechanical surface forces.

2.3 ACOUSTIC SCATTERING FROM PSS

Acoustic scattering from PSS has not been studied as well as the topics of vibration and acoustic radiation from PSS. Chertock, Hirsh, and Ogilvie²¹ obtained the flexural response of PSS to incident underwater explosion and tested the theory with experiments on a parallelepiped box-shell. Silbiger²² studied scattering from PSS due to an incident time-harmonic planewave by approximating the response from one resonant mode. Jones-Oliveira^{23,24} obtained the transient scattering from an axially incident plane wave, i.e., axisymmetric response, from a PSS that includes extensional and bending effects using expansions in terms of Legendre polynomials. She also obtained the resonances of two different PSS for the first eight flexural and longitudinal modes.

3. PROLATE SPHEROIDAL COORDINATE SYSTEM

In this report, the prolate spheroidal coordinate systems of Flammer²⁵ and Hanish, Baier, Van Buren, and King²⁶ have been employed. Basically, these coordinates represent ellipsoidal or hyperboloidal surfaces (see figure 3.1). The coordinate system transformations from cartesian coordinates are given by

$$\begin{aligned} x &= \frac{d}{2}[(1 - \eta^2)(\xi^2 - 1)]^{1/2} \cos \phi & -1 \leq \eta \leq 1, \\ y &= \frac{d}{2}[(1 - \eta^2)(\xi^2 - 1)]^{1/2} \sin \phi & 1 \leq \xi < \infty, \end{aligned} \quad (3.1)$$

and

$$z = \frac{d}{2}\eta\xi \quad 0 \leq \phi \leq 2\pi,$$

where d represents the interfocal distance of the generating ellipse.

In terms of cartesian coordinates, the prolate spheroidal coordinates are

$$\begin{aligned} \frac{x^2 + y^2}{\left(\frac{d}{2}\right)^2 (\xi^2 - 1)} + \frac{z^2}{\left(\frac{d}{2}\right)^2 \xi^2} &= 1 \quad (\text{ellipsoid surface for constant } \xi), \\ \frac{x^2 + y^2}{\left(\frac{d}{2}\right)^2 (1 - \eta^2)} - \frac{z^2}{\left(\frac{d}{2}\right)^2 \eta^2} &= -1 \quad (\text{hyperboloid surface for constant } \eta), \end{aligned} \quad (3.2)$$

and

$$\phi = \arctan (y/x) \quad (\text{half-plane surface for constant } \phi).$$

The ratio of minor to major axes of a prolate spheroid of constant ξ is

$$\alpha = \frac{\sqrt{\xi^2 - 1}}{\xi} \quad \xi > 1, \quad (3.3)$$

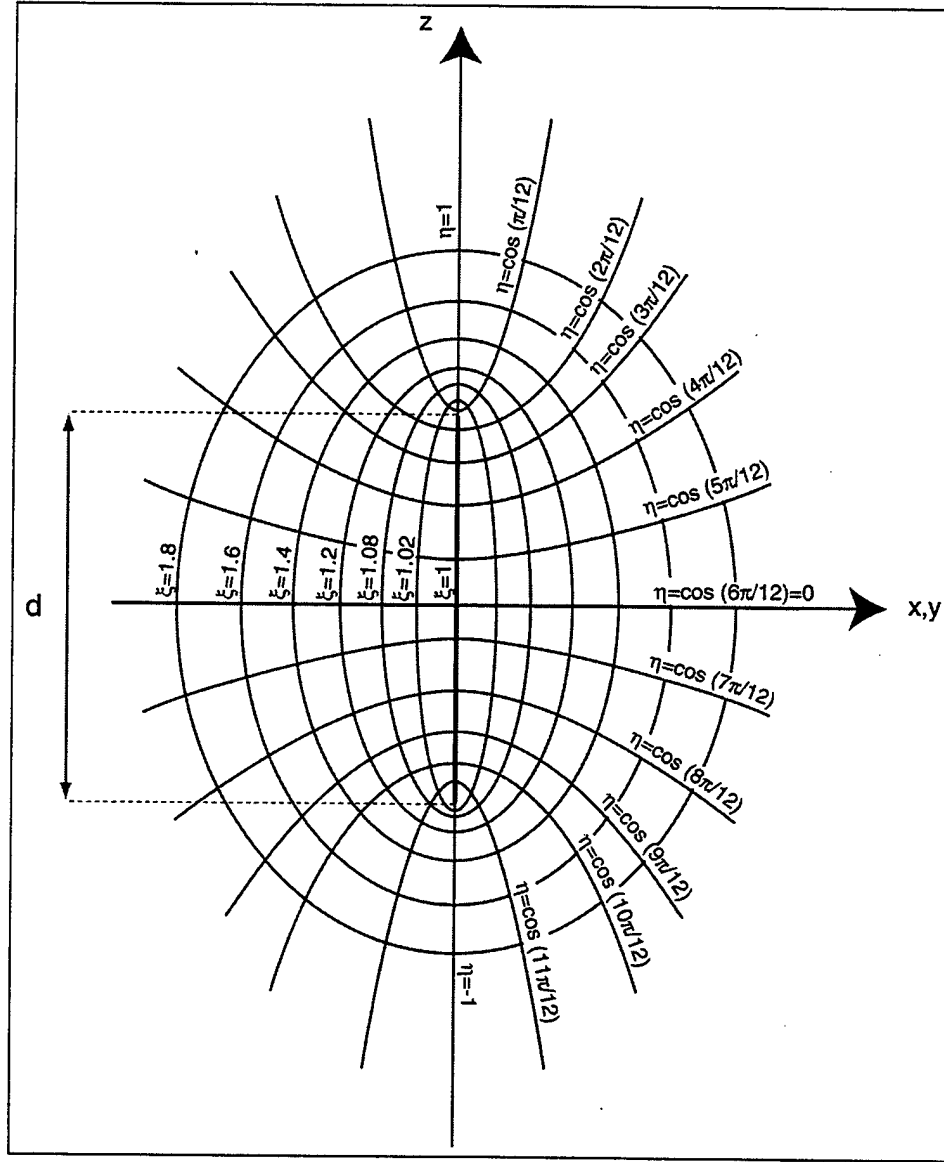


Figure 3.1. Prolate Spheroidal Coordinate System

and the eccentricity of the prolate spheroid is

$$e = \frac{1}{\xi}. \quad (3.4)$$

Thus, the prolate spheroid represents many ellipsoidal-shaped objects, e.g.,

$\xi \rightarrow 1$, $\alpha = 0$, a straight line of length d ,

$\xi \rightarrow \infty$, $\alpha = 1$, a sphere of constant radius $= (d\xi)/2$, $d \rightarrow 0$, $\xi \rightarrow \infty$,

$\eta = \pm 1$, semi-infinite lines along the z -axis,

$\eta = \text{constant}$, which represents a hyperboloid that makes an asymptotic angle $\theta = \arccos(\eta)$ with the z-axis, and

$\eta = 0$, the x-y plane.

Note that in the limit $\xi \rightarrow \infty$, $d \rightarrow 0$, the prolate spheroidal coordinates reduce to spherical coordinates, i.e., $d\xi/2 \rightarrow r$, $\eta \rightarrow \cos \theta$, as $d \rightarrow 0$ and $\xi \rightarrow \infty$.

The length and area elements are

$$ds^2 = dx^2 + dy^2 + dz^2 = h_\eta^2 d\eta^2 + h_\xi^2 d\xi^2 + h_\phi^2 d\phi^2,$$

where

$$h_\eta = \frac{d}{2} \left(\frac{\xi^2 - \eta^2}{1 - \eta^2} \right)^{1/2}, \quad h_\xi = \frac{d}{2} \left(\frac{\xi^2 - \eta^2}{\xi^2 - 1} \right)^{1/2}, \quad \text{and} \quad h_\phi = \frac{d}{2} [(1 - \eta^2)(\xi^2 - 1)]^{1/2}. \quad (3.5)$$

An area element on a surface $\xi = \text{constant}$ is defined as

$$dA = h_\eta h_\phi d\eta d\phi = \left(\frac{d}{2} \right)^2 \sqrt{(\xi^2 - \eta^2)(\xi^2 - 1)} d\eta d\phi, \quad (3.6)$$

and the gradient is defined by

$$\nabla\psi = \frac{\bar{e}_\xi}{(d/2)} \sqrt{\frac{\xi^2 - 1}{\xi^2 - \eta^2}} \frac{\partial\psi}{\partial\xi} + \frac{\bar{e}_\eta}{(d/2)} \sqrt{\frac{1 - \eta^2}{\xi^2 - \eta^2}} \frac{\partial\psi}{\partial\eta} + \frac{\bar{e}_\phi}{(d/2)} \sqrt{\frac{1}{(\xi^2 - 1)(1 - \eta^2)}} \frac{\partial\psi}{\partial\phi}. \quad (3.7)$$

The Helmholtz equation in prolate spheroidal coordinates is

$$(\nabla^2 + k^2)\psi = 0$$

$$\left[\frac{\partial}{\partial\eta} (1 - \eta^2) \frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi} (\xi^2 - 1) \frac{\partial}{\partial\xi} + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2}{\partial\phi^2} + c^2 (\xi^2 - \eta^2) \right] \psi = 0, \quad (3.8)$$

where the nondimensional wavenumber c is given by

$$c = \frac{1}{2} kd. \quad (3.9)$$

The Helmholtz equation is separable into prolate spheroidal wave functions, i.e.,

$$\psi_{mn} = S_{mn}(c, \eta) R_{mn}(c, \xi) \begin{bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{bmatrix}, \quad (3.10)$$

where the angular and radial wave functions $S_{mn}(c, \eta)$ and $R_{mn}(c, \xi)$ satisfy the following second-order differential equations:

$$\frac{d}{d\eta} \left[(1 - \eta^2) \frac{d}{d\eta} S_{mn}(c, \eta) \right] + \left[\lambda_{mn} - c^2 \eta^2 - \frac{m^2}{1 - \eta^2} \right] S_{mn}(c, \eta) = 0, \quad (3.11)$$

$$\frac{d}{d\xi} \left[(\xi^2 - 1) \frac{d}{d\xi} R_{mn}(c, \eta) \right] - \left[\lambda_{mn} - c^2 \xi^2 + \frac{m^2}{\xi^2 - 1} \right] R_{mn}(c, \eta) = 0, \quad (3.12)$$

where λ_{mn} is the separation constant.

Note that if $\xi \rightarrow \infty$, $d \rightarrow 0$, $c \rightarrow 0$, $\xi d/2 \rightarrow r$, $c\xi \rightarrow kr$, and $\eta \rightarrow \cos \theta$, then these equations reduce to the wave equation in spherical coordinates. Thus, $\lambda_{mn} \rightarrow \ell(\ell + 1)$, where $\ell = m, m + 1, m + 2, \dots$ and

$$S_{mn}(c, \eta) \rightarrow P_n^m(\eta) \text{ and } Q_n^m(\eta) \quad (\text{associated Legendre functions}),$$

$$R_{mn}(c, \xi) \rightarrow j_n(kr) \text{ and } y_n(kr) \text{ or } h_n^{(1)}(kr) \text{ and } h_n^{(2)}(kr) \quad (\text{spherical Bessel functions}).$$

It should be noted that the angular prolate spheroidal wave functions are dependent on wavenumber c unlike the angular spherical wave functions. Both of the prolate spheroidal wave functions can be expanded in terms of spherical wave functions.

4. DEFORMATIONS AND KINEMATICAL RELATIONS FOR CURVILINEAR THIN SHELLS

The theory of thin curvilinear shells depends on assumptions made in the theory of elastic media. Specifically, it is assumed that the shell is thin and that deformations are small. The shell's mid-surface is defined by general curvilinear orthogonal coordinates, α_1 and α_2 , where z is the coordinate normal to the mid-surface of the element (figure 4.1). The theories of thin shells have been developed in many books,²⁷⁻³¹ and special higher-order theories are presented in several research reports and archival papers.

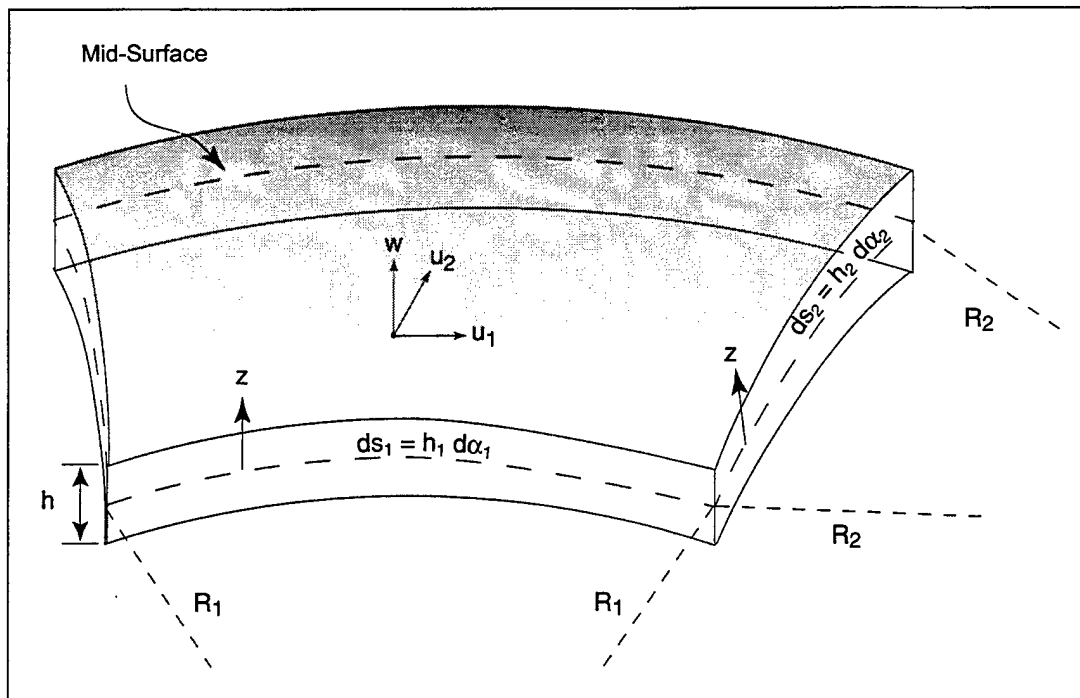


Figure 4.1. Deformed Element of a Shell

Let the displacements U_1 , U_2 , and W at any point (α_1, α_2, z) be given approximately by a Taylor series expansion in the thickness coordinate z , i.e.,

$$U_1(\alpha_1, \alpha_2, z) = u_1(\alpha_1, \alpha_2) + z \beta_1(\alpha_1, \alpha_2),$$

$$U_2(\alpha_1, \alpha_2, z) = u_2(\alpha_1, \alpha_2) + z \beta_2(\alpha_1, \alpha_2),$$

$$W(\alpha_1, \alpha_2, z) = w(\alpha_1, \alpha_2) + z w_1(\alpha_1, \alpha_2) + \frac{z^2}{2} w_2(\alpha_1, \alpha_2), \quad (4.1)$$

where z is the distance along the normal of the mid-surface, $-h/2 \leq z \leq h/2$.

The assumed displacement fields have in-plane displacements u_1 and u_2 and out-of plane displacement w , as well as thickness-shear deformations β_1 and β_2 and thickness-stretch deformations w_1 and w_2 . In this work, the thickness-stretch components w_1 and w_2 will be neglected in applications for mid to moderately high frequencies. With the neglect of w_1 and w_2 , there are five branches in the frequency–wavenumber spectrum (two acoustic and three optical branches) that extend the classical Love (thin-shell membrane-bending) theory to a much higher frequency range.

The six elastic strains at any point (α_1, α_2, z) are defined by Kraus²⁷ as

$$\epsilon_{11} = \frac{1}{1 + z/R_1} (\bar{\epsilon}_1 + zK_1),$$

$$\epsilon_{22} = \frac{1}{1 + z/R_2} (\bar{\epsilon}_2 + zK_2),$$

$$\epsilon_{33} = 0,$$

$$\gamma_{12} = \frac{1}{1 + z/R_1} (\bar{\gamma}_1 + z\tau_1) + \frac{1}{1 + z/R_2} (\bar{\gamma}_2 + z\tau_2),$$

$$\gamma_{13} = \frac{\bar{\mu}_1}{1 + z/R_1},$$

$$\gamma_{23} = \frac{\bar{\mu}_2}{1 + z/R_2}, \quad (4.2)$$

where the quantities with a bar represent the mid-surface normal and shear strains, and the other quantities reflect the linear deformation through the thickness. R_1 and R_2 are the principal radii of curvatures in the α_1 , α_2 directions, respectively.

The mid-surface normal and shear strains are given by

$$\begin{aligned}\bar{\epsilon}_1 &= \frac{1}{h_1} \frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{h_1 h_2} \frac{\partial h_1}{\partial \alpha_2} + \frac{w}{R_1}, \\ \bar{\epsilon}_2 &= \frac{1}{h_2} \frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{h_1 h_2} \frac{\partial h_2}{\partial \alpha_1} + \frac{w}{R_2}, \\ \bar{\gamma}_1 &= \frac{1}{h_1} \frac{\partial u_2}{\partial \alpha_1} - \frac{u_1}{h_1 h_2} \frac{\partial h_1}{\partial \alpha_2}, \\ \bar{\gamma}_2 &= \frac{1}{h_2} \frac{\partial u_1}{\partial \alpha_2} - \frac{u_2}{h_1 h_2} \frac{\partial h_2}{\partial \alpha_1}.\end{aligned}\tag{4.3}$$

The independent shear deformation components of the shear strains are given in terms of β_1 and β_2 as follows:

$$\begin{aligned}\tau_1 &= \frac{1}{h_1} \frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{h_1 h_2} \frac{\partial h_1}{\partial \alpha_2}, \\ \tau_2 &= \frac{1}{h_2} \frac{\partial \beta_2}{\partial \alpha_2} - \frac{\beta_2}{h_1 h_2} \frac{\partial h_2}{\partial \alpha_1},\end{aligned}\tag{4.4}$$

where τ_1 and τ_2 are the twists associated with the mid-surface, thus,

$$\begin{aligned}\bar{\mu}_1 &= \frac{1}{h_1} \frac{\partial w}{\partial \alpha_1} - \frac{u_1}{R_1} + \beta_1, \\ \bar{\mu}_2 &= \frac{1}{h_2} \frac{\partial w}{\partial \alpha_2} - \frac{u_2}{R_2} + \beta_2.\end{aligned}\tag{4.5}$$

Let the total twist τ and the total shear strain of the mid-surface $\bar{\gamma}_{12}$ be expressed by

$$\tau = \tau_1 + \tau_2 \quad \text{and} \quad \bar{\gamma}_{12} = \bar{\gamma}_1 + \bar{\gamma}_2.\tag{4.6}$$

The bending components of strain K_1 and K_2 are the changes of curvature of the mid-surface so that

$$\begin{aligned} K_1 &= \frac{1}{h_1} \frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{h_1 h_2} \frac{\partial h_1}{\partial \alpha_2}, \\ K_2 &= \frac{1}{h_2} \frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{h_1 h_2} \frac{\partial h_2}{\partial \alpha_1}. \end{aligned} \tag{4.7}$$

5. STRESS-STRAIN RELATIONS

The stress-strain relationships for an isotropic elastic shell can be written for the non-vanishing strains as

$$\begin{aligned}\sigma_{11} &= \frac{E}{1-\nu^2} [\epsilon_{11} + \nu \epsilon_{22}], \\ \sigma_{22} &= \frac{E}{1-\nu^2} [\epsilon_{22} + \nu \epsilon_{11}], \\ \tau_{12} &= G \gamma_{12}, \\ \tau_{13} &= G' \gamma_{13} = \kappa G \gamma_{13}, \\ \tau_{23} &= G' \gamma_{23} = \kappa G \gamma_{23},\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}G &= \frac{E}{2(1+\nu)}, \\ \kappa &= \pi^2 / 12, \\ G' &= \kappa G,\end{aligned}\tag{5.2}$$

where E is the Young's modulus, ν is the Poisson's ratio, G is the shear modulus, G' is the corrected shear modulus to compensate for the thickness-shear deformation, and κ is the correction factor given by Mindlin.^{32,33}

6. LAGRANGIAN OF THE DYNAMIC SYSTEM FOR A THIN SHELL

The equations of motion of the vibration of shells can be derived from the Lagrangian L defined as

$$L = -U + K + X, \quad (6.1)$$

where U is the strain energy of the shell, K is the kinetic energy of the shell, and X is the potential for the work done by external surface distributed forces.

The strain energy density \bar{U} per unit volume is defined by

$$\bar{U} = \frac{1}{2} [\sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \gamma_{12}\tau_{12} + \gamma_{13}\tau_{13} + \gamma_{23}\tau_{23}]. \quad (6.2)$$

Substituting the stress-strain relations in equation (5.1), the strain energy density is obtained as

$$\bar{U} = \frac{1}{2} \frac{E}{1-\nu^2} [\epsilon_{11}^2 + \epsilon_{22}^2 + 2\nu\epsilon_{11}\epsilon_{22} + \frac{1-\nu}{2} (\gamma_{12}^2 + \kappa\gamma_{13}^2 + \kappa\gamma_{23}^2)]. \quad (6.3)$$

The kinetic energy density per unit volume is defined by

$$\bar{K} = \frac{1}{2} \rho_s [\dot{U}_1^2 + \dot{U}_2^2 + \dot{W}^2], \quad (6.4)$$

where ρ_s is the density of the shell's material, and the dot indicates the partial derivative with respect to time $\partial/\partial t$.

A surface element dS at any surface z is

$$dS = (1 + z/R_1)(1 + z/R_2) d\bar{S}, \quad (6.5)$$

with the surface element at mid-surface $d\bar{S}$ defined as

$$d\bar{S} = h_1 h_2 d\alpha_1 d\alpha_2.$$

The total strain energy of a shell of variable thickness $h(\alpha_1, \alpha_2)$ is

$$U = \int_{\alpha_1} \int_{\alpha_2} \int_{-h/2}^{h/2} \bar{U}(\alpha_1, \alpha_2, z) dS dz. \quad (6.6)$$

The total kinetic energy of the shell is thus given by

$$K = \int_{\alpha_1} \int_{\alpha_2} \int_{-h/2}^{h/2} \bar{K}(\alpha_1, \alpha_2, z) dS dz. \quad (6.7)$$

The potential for external distributed force fields acting on the upper ($h/2$) and lower ($-h/2$) surfaces of the shell is the product of these force fields and the corresponding displacements at the upper and lower surfaces, integrated over the upper and lower surfaces.

Let $q_{u_1}^+$, $q_{u_2}^+$, and q_w^+ be the distributed forces per unit area in the 1, 2 and 3 directions, applied on the upper surface of the shell and let $q_{u_1}^-$, $q_{u_2}^-$, and q_w^- be the corresponding forces applied on the lower surface of the shell. Since the surface element dS is a function of z , then

$$\begin{aligned} dS^+ &= [1 + \frac{h}{2R_1}][1 + \frac{h}{2R_2}]d\bar{S}, \\ dS^- &= [1 - \frac{h}{2R_1}][1 - \frac{h}{2R_2}]d\bar{S}. \end{aligned} \quad (6.8)$$

The total work done by these distributed surface forces is the product of these forces and the corresponding displacements evaluated at the upper and lower surfaces, i.e.,

$$X = \int_{\alpha_1} \int_{\alpha_2} (q_{u_1}^+ U_1^+ + q_{u_2}^+ U_2^+ + q_w^+ W^+) dS^+ + \int_{\alpha_1} \int_{\alpha_2} (q_{u_1}^- U_1^- + q_{u_2}^- U_2^- + q_w^- W^-) dS^- \quad (6.9)$$

where the +/- superscripts indicate the upper/lower surfaces of the shell.

Since the dependence of the strains and velocities on z is known (see eqs. (4.1) and (4.2)), then one can perform the integration over z across the shell's thickness. To perform these integrations, expressions containing various terms in z must be approximated as follows for $\frac{h}{R} \ll 1$, valid up to the order $O(h^7/R^7)$:

$$\begin{aligned}
\int_{-h/2}^{h/2} (1+z/R_1)(1+z/R_2) dz &= h \left\{ 1 + \frac{h^2}{12R_1R_2} \right\}, \\
\int_{-h/2}^{h/2} (1+z/R_1)(1+z/R_2) z dz &= \frac{h^3}{12} \left\{ \frac{1}{R_1} + \frac{1}{R_2} \right\}, \\
\int_{-h/2}^{h/2} (1+z/R_1)(1+z/R_2) z^2 dz &= \frac{h^3}{12} \left\{ 1 + \frac{3}{20} \frac{h^2}{R_1R_2} \right\}, \\
\int_{-h/2}^{h/2} \frac{1+z/R_1}{1+z/R_2} dz &\approx h \left\{ 1 - \frac{h^2}{12R_2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) - \frac{h^5}{80R_2^3} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right\}, \\
\int_{-h/2}^{h/2} \frac{1+z/R_1}{1+z/R_2} z dz &\approx \frac{h^3}{12} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \left\{ 1 + \frac{3h^2}{20R_2^2} \right\}, \\
\int_{-h/2}^{h/2} \frac{1+z/R_1}{1+z/R_2} z^2 dz &\approx \frac{h^3}{12} \left\{ 1 - \frac{3h^2}{20R_2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right\}, \\
\int_{-h/2}^{h/2} \frac{1+z/R_1}{1+z/R_2} z^3 dz &\approx \frac{h^5}{80} \left(\frac{1}{R_1} - \frac{1}{R_2} \right), \\
\int_{-h/2}^{h/2} \frac{1+z/R_1}{1+z/R_2} z^4 dz &\approx \frac{h^5}{80}. \tag{6.10}
\end{aligned}$$

One may interchange R_1 with R_2 , and vice versa. Substituting the expression for the displacement field in equation (4.1), then the expression for the kinetic energy density in (6.4) becomes

$$\bar{K} = \frac{1}{2} \rho_s [(\dot{u}_1 + z\dot{\beta}_1)^2 + (\dot{u}_2 + z\dot{\beta}_2)^2 + \dot{w}^2].$$

Using equation (6.10), the integral of the kinetic energy over z only in equation (6.7) results in

$$\begin{aligned}
\int_{-h/2}^{h/2} \bar{K} dS dz = \frac{1}{2} \rho_2 h \left\{ \left[1 + \frac{h^2}{12 R_1 R_2} \right] [\dot{u}_1^2 + \dot{u}_2^2 + \dot{w}^2] + 2(\dot{u}_1 \dot{\beta}_1 + \dot{u}_2 \dot{\beta}_2) \frac{h^2}{12} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right. \\
\left. + \frac{h^2}{12} \left[1 + \frac{3}{20} \frac{h^2}{R_1 R_2} \right] [\dot{\beta}_1^2 + \dot{\beta}_2^2] \right\} d\bar{S}.
\end{aligned} \tag{6.11}$$

The expression for the total work done by external forces in equation (6.9) can also be simplified by substituting for the displacements and the surface elements at the upper/lower surface so that

$$U_1^\pm = u_1 \pm \frac{h}{2} \beta_1,$$

$$U_2^\pm = u_2 \pm \frac{h}{2} \beta_2,$$

and

$$W^\pm = w.$$

This results in the following expression for X:

$$X = \int_{\alpha_1} \int_{\alpha_2} (q_{u_1} u_1 + q_{u_2} u_2 + q_w w + m_{\beta_1} \beta_1 + m_{\beta_2} \beta_2) d\bar{S}, \tag{6.12}$$

where

$$\begin{aligned}
q_{u_1} &= (q_{u_1}^+ + q_{u_1}^-) \left(1 + \frac{h^2}{4 R_1 R_2} \right) + \frac{h}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) (q_{u_1}^+ - q_{u_1}^-), \\
q_{u_2} &= (q_{u_2}^+ + q_{u_2}^-) \left(1 + \frac{h^2}{4 R_1 R_2} \right) + \frac{h}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) (q_{u_2}^+ - q_{u_2}^-), \\
q_w &= (q_w^+ + q_w^-) \left(1 + \frac{h^2}{4 R_1 R_2} \right) + \frac{h}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) (q_w^+ - q_w^-), \\
m_{\beta_1} &= \frac{h}{2} \left\{ (q_{u_1}^+ - q_{u_1}^-) \left(1 + \frac{h^2}{4 R_1 R_2} \right) + \frac{h}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) (q_{u_1}^+ + q_{u_1}^-) \right\}, \\
m_{\beta_2} &= \frac{h}{2} \left\{ (q_{u_2}^+ - q_{u_2}^-) \left(1 + \frac{h^2}{4 R_1 R_2} \right) + \frac{h}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) (q_{u_2}^+ + q_{u_2}^-) \right\}.
\end{aligned} \tag{6.13}$$

Note that if the surface forces are applied equally at the two surfaces, i.e., if $q_{u_1}^+ = q_{u_2}^- = \bar{q}_{u_1} / 2$, $q_{u_2}^+ = q_{u_1}^- = \bar{q}_{u_2} / 2$, and $q_w^+ = q_w^- = \bar{q}_w / 2$, then the force fields are the resultants applied at the central surface of the shell, $z = 0$, i.e.:

$$q_{u_1} = \bar{q}_{u_1} \left(1 + \frac{h^2}{4R_1 R_2} \right),$$

$$q_{u_2} = \bar{q}_{u_2} \left(1 + \frac{h^2}{4R_1 R_2} \right)$$

$$q_w = \bar{q}_w \left(1 + \frac{h^2}{4R_1 R_2} \right),$$

$$m_{\beta_1} = \bar{q}_{u_1} \frac{h^2}{4} \left(\frac{1}{R_1} + \frac{1}{R_2} \right),$$

$$m_{\beta_2} = \bar{q}_{u_2} \frac{h^2}{4} \left(\frac{1}{R_1} + \frac{1}{R_2} \right),$$

where \bar{q} represent the equivalent force fields applied at the central surface, $z = 0$.

The integral of the strain energy density in equation (6.3) can be written as:

$$\begin{aligned} U = & \frac{1}{2} \frac{E}{1-\nu^2} \int_{\alpha_1} \int_{\alpha_2} \int_{-h/2}^{h/2} \left\{ (\epsilon_1 + zK_1)^2 A(z) + \frac{(\epsilon_2 + zK_2)^2}{A(z)} + 2\nu(\epsilon_1 + zK_1)(\epsilon_2 + zK_2) \right. \\ & + \frac{1-\nu}{2} \left[(\bar{\gamma}_1 + z\tau_1)^2 A(z) + \frac{(\bar{\gamma}_2 + z\tau_2)^2}{A(z)} + 2(\bar{\gamma}_1 + z\tau_1)(\bar{\gamma}_2 + z\tau_2) \right. \\ & \left. \left. + \kappa \left(\bar{\mu}_1^2 A(z) + \frac{\bar{\mu}_2^2}{A(z)} \right) \right] \right\} dz d\bar{S}, \end{aligned} \quad (6.14)$$

$$\text{where } A(z) = \frac{1+z/R_2}{1+z/R_1}.$$

Integrating the strain energy density over z and keeping terms up to $O(h^3)$ in the Taylor series expansion, one gets

$$U = \frac{1}{2} \frac{E}{1-\nu^2} \int_{\alpha_1} \int_{\alpha_2} \bar{\bar{U}} d\bar{S}$$

where

$$\begin{aligned} \bar{\bar{U}} = & h \left\{ \varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2 + \frac{1-\nu}{2} [(\bar{\gamma}_1 + \bar{\gamma}_2)^2 + \kappa(\bar{\mu}_1^2 + \bar{\mu}_2^2)] \right\} + \frac{h^3}{12} \left\{ (K_1^2 + K_2^2 + 2\nu K_1 K_2) \right. \\ & + \left[\frac{\varepsilon_1^2}{R_1} - \frac{\varepsilon_2^2}{R_2} - 2\varepsilon_1 K_1 + 2\varepsilon_2 K_2 \right] \left(\frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{1-\nu}{2} \left[\left[\frac{\bar{\gamma}_1^2}{R_1} - \frac{\bar{\gamma}_2^2}{R_2} - 2\bar{\gamma}_1 \tau_1 + 2\bar{\gamma}_2 \tau_2 \right] \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right. \\ & \left. \left. + (\tau_1 + \tau_2)^2 + \gamma \kappa \left(\frac{\bar{\mu}_1^2}{R_1} - \frac{\bar{\mu}_2^2}{R_2} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right] \right\}. \end{aligned} \quad (6.15)$$

Note that $\bar{\gamma}_1 + \bar{\gamma}_2 = \bar{\gamma}_{12}$. The constant γ in the last term of (6.15) was introduced by Naghdi³⁴ as $\gamma = 3/7$ to further correct the shear correction factor, κ .

7. GEOMETRIC AND KINEMATICAL EQUATIONS FOR A PSS OF CONSTANT THICKNESS

Consider a PSS of constant thickness whose mid-surface is defined by $\xi = a > 1$ and inter-focal distance d . A point on the shell's mid-surface is given by coordinates $\alpha_1 = \eta$ and $\alpha_2 = \phi$. The shell's length 2ℓ and maximum radius R are given by (see figure 7.1)

$$\text{Half length } \ell = da/2,$$

$$\text{Maximum radius } R = d\sqrt{a^2 - 1} / 2,$$

$$\text{Ratio of diameter/length} = \frac{\sqrt{a^2 - 1}}{a},$$

$$\text{Eccentricity } e = 1/a. \tag{7.1}$$

The metric coefficients of a surface element at the mid-surface of the shell, h_η and h_ϕ , and the surface element $d\bar{A}$ at any point η, ϕ are given by

$$h_\eta = \frac{d}{2} \frac{\sqrt{a^2 - \eta^2}}{\sqrt{1 - \eta^2}} \quad \text{and} \quad h_\phi = \frac{d}{2} \sqrt{a^2 - 1} \sqrt{1 - \eta^2}, \tag{7.2}$$

and

$$d\bar{S} = \left(\frac{d}{2}\right)^2 \sqrt{a^2 - 1} \sqrt{a^2 - \eta^2} d\eta d\phi. \tag{7.3}$$

The principal radii of curvatures at any point (η, ϕ) are

$$R_\eta = R_1 = \frac{d}{2a} \frac{1}{\sqrt{a^2 - 1}} (a^2 - \eta^2)^{3/2},$$

$$R_\phi = R_2 = \frac{d}{2a} \sqrt{a^2 - 1} \sqrt{a^2 - \eta^2}. \tag{7.4}$$

Letting

$$\begin{array}{llll} u_1 = u, & u_2 = v, & \beta_1 = \beta_\eta, & \beta_2 = \beta_\phi, \\ R_1 = R_\eta, & R_2 = R_\phi, & h_1 = h_\eta, & h_2 = h_\phi, \end{array}$$

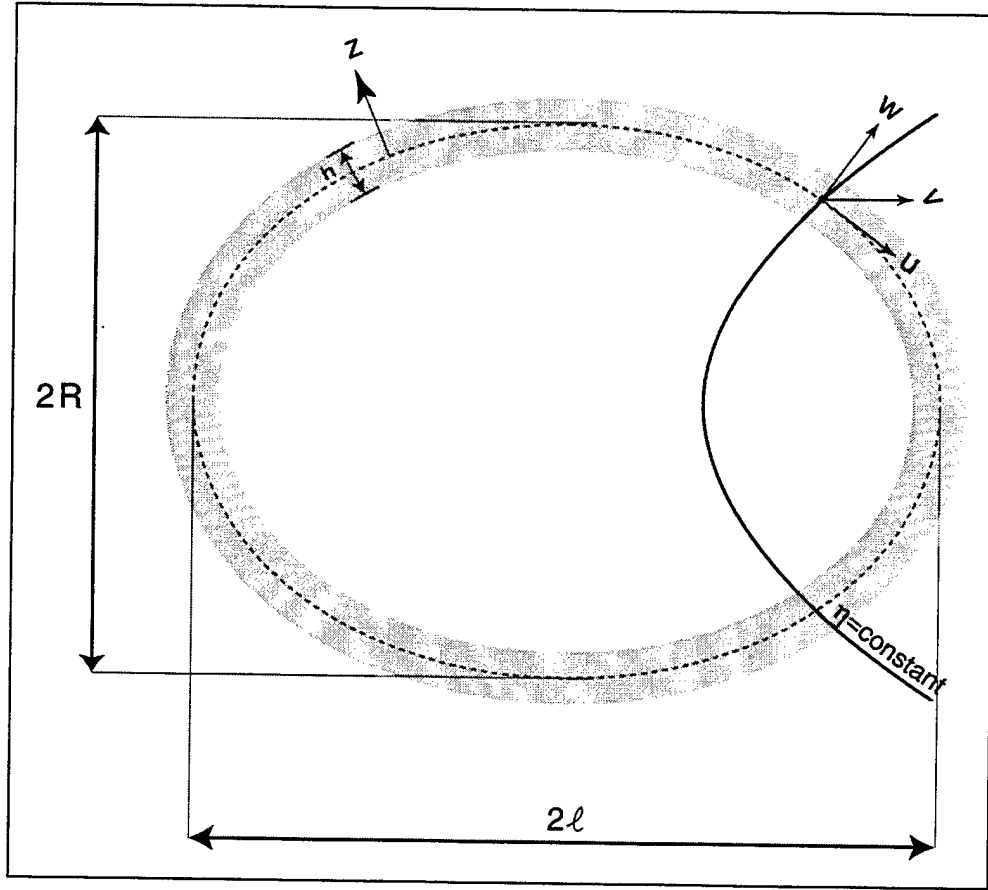


Figure 7.1. Geometry of a PSS of Constant Thickness

then the kinematical variables for a PSS shell become

$$\begin{aligned}
 \epsilon_1 = \epsilon_\eta &= \frac{1}{h_\eta} \frac{\partial u}{\partial \eta} + \frac{w}{R_\eta}, \\
 \epsilon_2 = \epsilon_\phi &= \frac{1}{h_\phi} \frac{\partial v}{\partial \phi} + \frac{u}{h_\eta h_\phi} \frac{\partial h_\phi}{\partial \eta} + \frac{w}{R_\phi}, \\
 \bar{\gamma}_1 = \bar{\gamma}_\eta &= \frac{1}{h_\eta} \frac{\partial v}{\partial \eta}, \\
 \bar{\gamma}_2 = \bar{\gamma}_\phi &= \frac{1}{h_\phi} \frac{\partial u}{\partial \phi} - \frac{v}{h_\eta h_\phi} \frac{\partial h_\phi}{\partial \eta}, \\
 \bar{\gamma}_1 + \bar{\gamma}_2 = \bar{\gamma}_{12} = \bar{\gamma}_{\eta\phi} &= \frac{1}{h_\eta} \frac{\partial v}{\partial \eta} - \frac{v}{h_\eta h_\phi} \frac{\partial h_\phi}{\partial \eta} + \frac{1}{h_\phi} \frac{\partial u}{\partial \phi},
 \end{aligned} \tag{7.5}$$

and

$$\begin{aligned}K_1 &= K_\eta = \frac{1}{h_\eta} \frac{\partial \beta_\eta}{\partial \eta}, \\K_2 &= K_\phi = \frac{1}{h_\phi} \frac{\partial \beta_\phi}{\partial \phi} + \frac{1}{h_\eta h_\phi} \frac{\partial h_\phi}{\partial \eta} \beta_\eta, \\ \bar{\mu}_1 &= \bar{\mu}_\eta = \frac{1}{h_\eta} \frac{\partial w}{\partial \eta} - \frac{u}{R_\eta} + \beta_\eta, \\ \bar{\mu}_2 &= \bar{\mu}_\phi = \frac{1}{h_\phi} \frac{\partial w}{\partial \phi} - \frac{v}{R_\phi} + \beta_\phi.\end{aligned}\tag{7.6}$$

8. DERIVATION OF THE EQUATIONS OF MOTION

The equations of motion of the shell are derived from the Lagrangian. Let a function L be

$$L = L(x_j, y_k, \frac{\partial y_k}{\partial x_j}, \frac{\partial^2 y_k}{\partial x_j \partial x_\ell}) = L(x_j, y_k, y_{k,j}, y_{k,j\ell}), \quad k = 1, 2, 3, 4, 5; \quad j, \ell = 1, 2, 3, \quad (8.1)$$

where x_j are the independent variables, y_k are the dependant variables, and the comma in the sub-indices indicates partial differentiation. The Lagrangian L is given as an integral, i.e.,

$$F = \int_{\alpha_1} \int_{\alpha_2} L dS dt = \int_{\alpha_1} \int_{\alpha_2} (-U + K + X) d\bar{S} dt. \quad (8.2)$$

The partial differential equations of motion are then given by

$$\begin{aligned} \frac{\partial L}{\partial y_k} - \frac{\partial}{\partial x_1} \left(\frac{\partial L}{\partial y_{k,1}} \right) - \frac{\partial}{\partial x_2} \left(\frac{\partial L}{\partial y_{k,2}} \right) - \frac{\partial}{\partial x_3} \left(\frac{\partial L}{\partial y_{k,3}} \right) + \frac{\partial^2}{\partial x_1^2} \left(\frac{\partial L}{\partial y_{k,11}} \right) + \frac{\partial^2}{\partial x_2^2} \left(\frac{\partial L}{\partial y_{k,22}} \right) \\ + \frac{\partial^2}{\partial x_3^2} \left(\frac{\partial L}{\partial y_{k,33}} \right) + \frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{\partial L}{\partial y_{k,12}} \right) + \frac{\partial^2}{\partial x_2 \partial x_3} \left(\frac{\partial L}{\partial y_{k,23}} \right) + \frac{\partial^2}{\partial x_3 \partial x_1} \left(\frac{\partial L}{\partial y_{k,31}} \right) = 0, \end{aligned} \quad (8.3)$$

where $k = 1, 2, 3, 4, 5$. Letting

$$\begin{aligned} x_1 = \eta, \quad x_2 = \phi, \quad x_3 = t, \\ y_1 = u, \quad y_2 = v, \quad y_3 = w, \quad y_4 = \beta_\eta, \quad \text{and} \quad y_5 = \beta_\phi, \end{aligned}$$

then there are five equations of motion on the five independent kinematical variables.

One can obtain the five coupled partial differential equations by applying the calculus of variations, equation (8.3), on the Lagrangian L . Since the Lagrangian is a very long expression, the application of equation (8.3) is performed for each component of L , i.e., K , X , and U .

To simplify the algebraic manipulations, one can multiply \mathbf{L} by $(1-v^2)/(Eh)$. To non-dimensionalize the independent variables, let

$$\bar{u} = u / \ell \quad \bar{v} = v / \ell \quad \bar{w} = w / \ell, \quad (8.4)$$

and

$$c_p^2 = \frac{E}{\rho_s (1-v^2)}, \quad (8.5)$$

where

$$\ell = \frac{da}{2} = \text{half-length of the shell, and } \epsilon = \frac{h^2}{12\ell^2} \text{ is the flexure coefficient.} \quad (8.6)$$

8.1 EXTERNAL WORK

The external work X has terms with y_k only, so that the application of (8.3) results in the following terms of the type $\partial X / \partial y_k$:

$$\text{Eq 1: } \frac{1-v^2}{Eh\ell} q_u h_\eta h_\phi,$$

$$\text{Eq 2: } \frac{1-v^2}{Eh\ell} q_v h_\eta h_\phi,$$

$$\text{Eq 3: } \frac{1-v^2}{Eh\ell} q_w h_\eta h_\phi, \quad (8.7)$$

$$\text{Eq 4: } \frac{1-v^2}{Eh\ell^2} m_{\beta_\eta} h_\eta h_\phi,$$

$$\text{Eq 5: } \frac{1-v^2}{Eh\ell^2} m_{\beta_\phi} h_\eta h_\phi.$$

8.2 KINETIC ENERGY

The inertial terms are determined by using (8.3). The kinetic energy has terms of the type $y_{k,t}$ so that the application of equations (8.3) results in differentials of the type $-\frac{\partial}{\partial t} \left(\frac{\partial K}{\partial y_{k,t}} \right)$, i.e.,

$$\begin{aligned}
\text{Eq 1: } & -\frac{1}{c_p^2} \left[\ddot{u} \left(1 + \frac{h^2}{12 R_\eta R_\phi} \right) + \frac{h^2}{12 \ell} \left(\frac{1}{R_\eta} + \frac{1}{R_\phi} \right) \ddot{\beta}_\eta \right] h_\eta h_\phi, \\
\text{Eq 2: } & -\frac{1}{c_p^2} \left[\ddot{v} \left(1 + \frac{h^2}{12 R_\eta R_\phi} \right) + \frac{h^2}{12 \ell} \left(\frac{1}{R_\eta} + \frac{1}{R_\phi} \right) \ddot{\beta}_\phi \right] h_\eta h_\phi, \\
\text{Eq 3: } & -\frac{1}{c_p^2} \left[\ddot{w} \left(1 + \frac{h^2}{12 R_\eta R_\phi} \right) \right] h_\eta h_\phi, \\
\text{Eq 4: } & -\frac{1}{c_p^2} \frac{h^2}{12 \ell} \left[\frac{1}{\ell} \ddot{\beta}_\eta \left(1 + \frac{3h^2}{20 R_\eta R_\phi} \right) + \ddot{u} \left(\frac{1}{R_\eta} + \frac{1}{R_\phi} \right) \right] h_\eta h_\phi, \\
\text{Eq 5: } & -\frac{1}{c_p^2} \frac{h^2}{12 \ell} \left[\frac{1}{\ell} \ddot{\beta}_\phi \left(1 + \frac{3h^2}{20 R_\eta R_\phi} \right) + \ddot{v} \left(\frac{1}{R_\eta} + \frac{1}{R_\phi} \right) \right] h_\eta h_\phi.
\end{aligned} \tag{8.8}$$

8.3 STRAIN ENERGY

The strain energy has terms with partial differentials of up to second-order in η and ϕ . Applying (8.3) to the expressions for U in (6.15) would result in five coupled partial differential equations. These can be written symbolically as

$$\begin{aligned}
\text{Eq 1: } & L_{uu} \bar{u} + L_{uv} \bar{v} + L_{uw} \bar{w} + L_{u\beta_\eta} \beta_\eta + L_{u\beta_\phi} \beta_\phi, \\
\text{Eq 2: } & L_{vu} \bar{u} + L_{vv} \bar{v} + L_{vw} \bar{w} + L_{v\beta_\eta} \beta_\eta + L_{v\beta_\phi} \beta_\phi, \\
\text{Eq 3: } & L_{wu} \bar{u} + L_{wv} \bar{v} + L_{ww} \bar{w} + L_{w\beta_\eta} \beta_\eta + L_{w\beta_\phi} \beta_\phi, \\
\text{Eq 4: } & L_{\beta_\eta u} \bar{u} + L_{\beta_\eta v} \bar{v} + L_{\beta_\eta w} \bar{w} + L_{\beta_\eta \beta_\eta} \beta_\eta + L_{\beta_\eta \beta_\phi} \beta_\phi, \\
\text{Eq 5: } & L_{\beta_\phi u} \bar{u} + L_{\beta_\phi v} \bar{v} + L_{\beta_\phi w} \bar{w} + L_{\beta_\phi \beta_\eta} \beta_\eta + L_{\beta_\phi \beta_\phi} \beta_\phi.
\end{aligned} \tag{8.9}$$

The partial differential operators (L_{ij}) will be defined in the proceeding subsections. To simplify the expressions, the following variables will be used:

$$\begin{aligned}
A &= a^2 - 1 & B &= 1 - \eta^2 & C &= a^2 - \eta^2, \\
D &= \frac{\sqrt{a^2 - 1}}{\sqrt{a^2 - \eta^2}} & F &= \frac{1 - \nu}{2} & \gamma &= 3/7, & \kappa &= \pi^2/12.
\end{aligned}$$

8.4 EQUATIONS OF MOTION FOR NONAXISYMMETRIC VIBRATIONS

8.4.1 First Equation

$$\begin{aligned}
L_{uu}\bar{u} &= DB \frac{\partial^2 \bar{u}}{\partial \eta^2} - \eta D(1 + D^2) \frac{\partial \bar{u}}{\partial \eta} + \frac{F}{DB} \frac{\partial^2 \bar{u}}{\partial \phi^2} + \left[-\eta^2 \frac{D}{B} - \nu a^2 \frac{D}{C} - F \kappa a^2 \frac{D^3}{C} \right] \bar{u} \\
&+ \varepsilon \left\{ -a^4 \frac{BD}{C^3} \left[B \frac{\partial^2 \bar{u}}{\partial \eta^2} + \eta(3 - 7D^2) \frac{\partial \bar{u}}{\partial \eta} \right] + F \frac{a^4 D^3}{A^3} \frac{\partial^2 \bar{u}}{\partial \phi^2} - \left[a^4 \eta^2 \frac{D}{AC^2} - F \gamma \kappa a^6 \frac{D^3 B}{C^4} \right] \bar{u} \right\}, \quad (8.10)
\end{aligned}$$

$$L_{uv}\bar{v} = (1 + F) \frac{\eta}{B} \frac{\partial \bar{v}}{\partial \phi} + (\nu + F) \frac{\partial^2 \bar{v}}{\partial \eta \partial \phi} + \varepsilon \left\{ \frac{a^4 \eta}{AC^2} (1 + F) \frac{\partial \bar{v}}{\partial \phi} \right\}, \quad (8.11)$$

$$\begin{aligned}
L_{uw}\bar{w} &= \frac{aD\sqrt{B}}{\sqrt{C}} \left[(1 + F\kappa) D^2 + \nu \right] \frac{\partial \bar{w}}{\partial \eta} + a \frac{\eta\sqrt{B}}{C^{5/2}} (4A + B) \bar{w} \\
&+ \varepsilon \left\{ a^5 A \frac{\sqrt{B}}{C^{5/2}} \left[-\frac{B}{C^2} (1 + F\gamma\kappa) \frac{\partial \bar{w}}{\partial \eta} + \eta \left[\frac{1}{A^2} - \frac{6}{C^2} + 9 \frac{A}{C^3} \right] \bar{w} \right] \right\}, \quad (8.12)
\end{aligned}$$

$$\begin{aligned}
L_{u\beta_\eta}\beta_\eta &= F\kappa D^2 \beta_\eta + \varepsilon \left\{ a^2 \frac{B^2}{C^2} \frac{\partial^2 \beta_\eta}{\partial \eta^2} - 4a^2 \eta \frac{AB}{C^3} \frac{\partial \beta_\eta}{\partial \eta} - F \frac{a^2}{AC} \frac{\partial^2 \beta_\eta}{\partial \phi^2} \right. \\
&\left. + \left[\frac{a^2 \eta^2}{C^2} - F \gamma \kappa \frac{a^4 AB}{C^4} \right] \beta_\eta \right\}, \quad (8.13)
\end{aligned}$$

$$L_{u\beta_\phi}\beta_\phi = \varepsilon \left\{ -(1 + F) a^2 \eta \frac{D}{AC} \frac{\partial \beta_\phi}{\partial \phi} \right\}. \quad (8.14)$$

8.4.2 Second Equation

$$L_{vu}\bar{u} = (v + F)\frac{\partial^2 \bar{u}}{\partial \eta \partial \phi} - \frac{\eta}{B}(1 + F)\frac{\partial \bar{u}}{\partial \phi} + \varepsilon \left\{ -(1 + F)\frac{a^4 \eta}{AC^2} \frac{\partial \bar{u}}{\partial \phi} \right\}, \quad (8.15)$$

$$\begin{aligned} L_{vv}\bar{v} = & FDB\frac{\partial^2 \bar{v}}{\partial \eta^2} - FD(1 + D^2)\eta\frac{\partial \bar{v}}{\partial \eta} + \frac{1}{DB}\frac{\partial^2 \bar{v}}{\partial \phi^2} + F\left[a^2\frac{D}{C} - \eta^2\frac{D}{B} - \kappa a^2\frac{D}{A}\right]\bar{v} \\ & + \varepsilon \left\{ -Fa^4\frac{BD}{C^3}\left[B\frac{\partial^2 \bar{v}}{\partial \eta^2} + \eta(3 - 7D^2)\frac{\partial \bar{v}}{\partial \eta}\right] + a^4\frac{D^3}{A^3}\frac{\partial^2 \bar{v}}{\partial \phi^2} - \frac{Fa^4D}{C^2A}\left[\eta^2 + \gamma\kappa\frac{a^2B}{A}\right]\bar{v} \right\}, \end{aligned} \quad (8.16)$$

$$L_{vw}\bar{w} = \frac{a}{\sqrt{AB}}\left[(1 + F\kappa) + vD^2\right]\frac{\partial \bar{w}}{\partial \phi} + \varepsilon \left\{ (1 + F\gamma\kappa)\frac{a^5\sqrt{B}}{A^{3/2}C^2} \frac{\partial \bar{w}}{\partial \phi} \right\}, \quad (8.17)$$

$$L_{v\beta_\eta}\beta_\eta = \varepsilon \left\{ (1 + F)\eta a^2\frac{D^3}{A^2} \frac{\partial \beta_\eta}{\partial \phi} \right\}, \quad (8.18)$$

$$\begin{aligned} L_{v\beta_\phi}\beta_\phi = & F\kappa\beta_\phi + \varepsilon \left\{ -4F\frac{a^2AB}{C^3}\eta\frac{\partial \beta_\phi}{\partial \eta} - \frac{a^2}{AC}\frac{\partial^2 \beta_\phi}{\partial \phi^2} + Fa^2\frac{B^2}{C^2}\frac{\partial^2 \beta_\phi}{\partial \eta^2} \right. \\ & \left. + F\left(\frac{\eta^2 a^2}{C^2} + \gamma\kappa\frac{a^4B}{AC^2}\right)\beta_\phi \right\}. \end{aligned} \quad (8.19)$$

8.4.3 Third Equation

$$\begin{aligned} L_{wu}\bar{u} = & -a\frac{\sqrt{B}}{C^{3/2}}\left[A(1 + F\kappa) + vC\right]\frac{\partial \bar{u}}{\partial \eta} + \frac{a\eta}{\sqrt{BC}}\left[1 + (v + F\kappa)D^2 - 3F\kappa\frac{AB}{C^2}\right]\bar{u} \\ & + \varepsilon \left\{ a^5\frac{AB^{3/2}}{C^{9/2}}(1 + F\kappa\gamma)\frac{\partial \bar{u}}{\partial \eta} + \eta\frac{a^5\sqrt{B}}{AC^{5/2}}\left[1 + 3F\kappa\gamma D^4(2 - 3D^2)\right]\bar{u} \right\}, \end{aligned} \quad (8.20)$$

$$L_{wv}\bar{v} = -\frac{a}{\sqrt{AB}}(1 + F\kappa + vD^2)\frac{\partial \bar{v}}{\partial \phi} + \varepsilon \left\{ -(1 + F\kappa\gamma)\frac{a^5\sqrt{B}}{A^{3/2}C^2} \frac{\partial \bar{v}}{\partial \phi} \right\}, \quad (8.21)$$

$$L_{ww}\bar{w} = F\kappa D \left[B \frac{\partial^2 \bar{w}}{\partial \eta^2} - (1 + D^2) \eta \frac{\partial \bar{w}}{\partial \eta} + \frac{1}{D^2 B} \frac{\partial^2 \bar{w}}{\partial \phi^2} \right] - \frac{a^2 D}{A} [1 + D^4 + 2\nu D^2] \bar{w} \\ + \varepsilon \left\{ -F\kappa\gamma \frac{a^4 DB}{C^3} \left[B \frac{\partial^2 \bar{w}}{\partial \eta^2} + (3 - 7D^2) \eta \frac{\partial \bar{w}}{\partial \eta} \right] + F\kappa\gamma \frac{a^4 D^3}{A^3} \frac{\partial^2 \bar{w}}{\partial \phi^2} + a^6 \frac{D^3 B}{C} \left[\frac{1}{C^3} - \frac{1}{A^3} \right] \bar{w} \right\}, \quad (8.22)$$

$$L_{w\beta_\eta}\beta_\eta = F\kappa \frac{\sqrt{A}}{a\sqrt{B}} \left[B \frac{\partial \beta_\eta}{\partial \eta} - \eta \beta_\eta \right] + \varepsilon \left\{ -a^3 \frac{\sqrt{BA}}{C^3} \left[B(F\kappa\gamma + 1) \frac{\partial \beta_\eta}{\partial \eta} \right] \right. \\ \left. - \frac{a^3 \sqrt{B}}{C^2 \sqrt{A}} \eta [1 + 3F\kappa\gamma D^2 (1 - 2D^2) \beta_\eta] \right\}, \quad (8.23)$$

$$L_{w\beta_\phi}\beta_\phi = F\kappa \frac{\sqrt{C}}{a\sqrt{B}} \frac{\partial \beta_\phi}{\partial \phi} + \varepsilon \left\{ [1 + F\kappa\gamma] \frac{a^3 \sqrt{B}}{AC^{3/2}} \frac{\partial \beta_\phi}{\partial \phi} \right\}. \quad (8.24)$$

8.4.4 Fourth Equation

$$L_{\beta_\eta u}\bar{u} = F\kappa D^2 \bar{u} + \varepsilon \left\{ a^2 \frac{B^2}{C^2} \frac{\partial^2 \bar{u}}{\partial \eta^2} - 4 \frac{a^2 AB}{C^3} \eta \frac{\partial \bar{u}}{\partial \eta} - F \frac{a^2}{AC} \frac{\partial^2 \bar{u}}{\partial \phi^2} + \left[\frac{a^2 \eta^2}{C^2} - F\kappa\gamma \frac{a^4 AB}{C^4} \right] \bar{u} \right\}, \quad (8.25)$$

$$L_{\beta_\eta v}\bar{v} = \varepsilon \left\{ -(1 + F) \frac{a^2 \eta}{C^{3/2} \sqrt{A}} \frac{\partial \bar{v}}{\partial \phi} \right\}, \quad (8.26)$$

$$L_{\beta_\eta w}\bar{w} = -F\kappa \frac{\sqrt{AB}}{a} \frac{\partial \bar{w}}{\partial \eta} + \varepsilon \left\{ \frac{a^3 \sqrt{B}}{C^3 \sqrt{A}} \left[(1 + F\kappa\gamma) AB \frac{\partial \bar{w}}{\partial \eta} + \eta C (-1 + 3D^2 (1 - 2D^2)) \bar{w} \right] \right\}, \quad (8.27)$$

$$L_{\beta_\eta \beta_\eta} \beta_\eta = -F\kappa \frac{DC}{a^2} \beta_\eta + \varepsilon \left\{ D \left[B \frac{\partial^2 \beta_\eta}{\partial \eta^2} - \eta (1 + D^2) \frac{\partial \beta_\eta}{\partial \eta} \right] + \frac{F}{DB} \frac{\partial^2 \beta_\eta}{\partial \phi^2} \right. \\ \left. + D \left(-\frac{\eta^2}{B} - \nu \frac{a^2}{C} + F\kappa\gamma \frac{a^2 B}{C^2} \right) \beta_\eta \right\}, \quad (8.28)$$

$$L_{\beta_\eta \beta_\phi} \beta_\phi = \varepsilon \left\{ (v + F) \frac{\partial^2 \beta_\phi}{\partial \eta \partial \phi} + (1 + F) \frac{\eta}{B} \frac{\partial \beta_\phi}{\partial \phi} \right\}. \quad (8.29)$$

8.4.5 Fifth Equation

$$L_{\beta_\phi u} \bar{u} = \varepsilon \left\{ (1 + F) \frac{a^2 \eta D}{AC} \frac{\partial \bar{u}}{\partial \phi} \right\}, \quad (8.30)$$

$$L_{\beta_\phi v} \bar{v} = F \kappa \bar{v} + \varepsilon \left\{ \frac{Fa^2 B}{C^2} \left[B \frac{\partial^2 \bar{v}}{\partial \eta^2} - 4 \frac{A}{C} \eta \frac{\partial \bar{v}}{\partial \eta} \right] - \frac{a^2}{AC} \frac{\partial^2 \bar{v}}{\partial \phi^2} + \frac{F}{C^2} \left[a^2 \eta^2 + \kappa \gamma \frac{a^4 B}{A} \right] \bar{v} \right\}, \quad (8.31)$$

$$L_{\beta_\phi w} \bar{w} = -F \kappa \frac{\sqrt{C}}{a \sqrt{B}} \frac{\partial \bar{w}}{\partial \phi} + \varepsilon \left\{ -(1 + F \kappa \gamma) \frac{a^3 \sqrt{B}}{AC^{3/2}} \frac{\partial \bar{w}}{\partial \phi} \right\}, \quad (8.32)$$

$$L_{\beta_\phi \beta_\eta} \beta_\eta = \varepsilon \left\{ (v + F) \frac{\partial^2 \beta_\eta}{\partial \eta \partial \phi} - (1 + F) \frac{\eta}{B} \frac{\partial \beta_\eta}{\partial \phi} \right\}, \quad (8.33)$$

$$\begin{aligned} L_{\beta_\phi \beta_\phi} \beta_\phi = & -F \kappa \frac{\sqrt{AC}}{a^2} \beta_\phi + \varepsilon \left\{ FDB \frac{\partial^2 \beta_\phi}{\partial \eta^2} - F \eta D (1 + D^2) \frac{\partial \beta_\phi}{\partial \eta} + \frac{1}{DB} \frac{\partial^2 \beta_\phi}{\partial \phi^2} \right. \\ & \left. + Fa^2 D \left[\frac{1}{C} - \frac{\eta^2}{a^2 B} - \kappa \gamma \frac{B}{AC} \right] \beta_\phi \right\}. \end{aligned} \quad (8.34)$$

8.4.6 Final Nondimensional Form of the Equations

$$\text{Eq 1: } L_{uu} \bar{u} + L_{uv} \bar{v} + L_{uw} \bar{w} + L_{u\beta_\eta} \beta_\eta + L_{u\beta_\phi} \beta_\phi$$

$$= \frac{\ell^2}{c_p^2} \left\{ (1 + \varepsilon \frac{a^4}{C^2}) \frac{\sqrt{AC}}{a^2} \ddot{\bar{u}} + \varepsilon (1 + D^2) \ddot{\beta}_\eta \right\} - \frac{1 - v^2}{Eh} \ell \frac{\sqrt{AC}}{a^2} q_u, \quad (8.35)$$

$$\begin{aligned}
\text{Eq 2: } & L_{vu}\bar{u} + L_{vv}\bar{v} + L_{vw}\bar{w} + L_{v\beta_\eta}\beta_\eta + L_{v\beta_\phi}\beta_\phi \\
& = \frac{\ell^2}{c_p^2} \left\{ \left(1 + \varepsilon \frac{a^4}{C^2}\right) \frac{\sqrt{AC}}{a^2} \ddot{\bar{v}} + \varepsilon(1 + D^2) \ddot{\beta}_\phi \right\} - \frac{1 - v^2}{Eh} \ell \frac{\sqrt{AC}}{a^2} q_v,
\end{aligned} \tag{8.36}$$

$$\begin{aligned}
\text{Eq 3: } & L_{wu}\bar{u} + L_{wv}\bar{v} + L_{ww}\bar{w} + L_{w\beta_\eta}\beta_\eta + L_{w\beta_\phi}\beta_\phi \\
& = \frac{\ell^2}{c_p^2} \frac{\sqrt{AC}}{a^2} \left(1 + \varepsilon \frac{a^4}{C^2}\right) \ddot{\bar{w}} - \frac{1 - v^2}{Eh} \ell \frac{\sqrt{AC}}{a^2} q_w,
\end{aligned} \tag{8.37}$$

$$\begin{aligned}
\text{Eq 4: } & L_{\beta_\eta u}\bar{u} + L_{\beta_\eta v}\bar{v} + L_{\beta_\eta w}\bar{w} + L_{\beta_\eta \beta_\eta}\beta_\eta + L_{\beta_\eta \beta_\phi}\beta_\phi \\
& = \frac{\ell^2}{c_p^2} \varepsilon \left\{ \frac{\sqrt{AC}}{a^2} \ddot{\beta}_\eta + (1 + D^2) \ddot{\bar{u}} \right\} - \frac{1 - v^2}{Eh} \frac{\sqrt{AC}}{a^2} m_{\beta_\eta},
\end{aligned} \tag{8.38}$$

$$\begin{aligned}
\text{Eq 5: } & L_{\beta_\phi u}\bar{u} + L_{\beta_\phi v}\bar{v} + L_{\beta_\phi w}\bar{w} + L_{\beta_\phi \beta_\eta}\beta_\eta + L_{\beta_\phi \beta_\phi}\beta_\phi \\
& = \frac{\ell^2}{c_p^2} \varepsilon \left\{ \frac{\sqrt{AC}}{a^2} \ddot{\beta}_\phi + (1 + D^2) \ddot{\bar{v}} \right\} - \frac{1 - v^2}{Eh} \frac{\sqrt{AC}}{a^2} m_{\beta_\phi}.
\end{aligned} \tag{8.39}$$

8.5 EQUATIONS OF MOTION FOR AXISYMMETRIC VIBRATIONS

For axisymmetric motion of PSS, $\partial/\partial\phi = 0$, and hence the five coupled equations decouple into three coupled equations on u , w , and β_η and two coupled equations on v and β_ϕ . It can be seen that the expressions in (8.11), (8.14) in the first equation, (8.21) and (8.24) in the third equation and (8.26) and (8.29) in the fourth equation vanish, rendering the first, third, and fourth equations dependent on u , w , and β_η exclusively. Furthermore, for axisymmetric motion, expressions in (8.15), (8.17), and (8.18) in the second equation, (8.30), (8.32), and (8.33) in the fifth equation vanish. This means that the second and fifth equations are exclusively dependent on v and β_ϕ .

9. REDUCTION TO SPHERICAL SHELLS

The equations of motion of spherical shells with shear deformations and rotatory inertias can be recovered from section 8 by taking the limits, $d \rightarrow 0$, $a \rightarrow \infty$, and $\left(\frac{da}{2}\right) = \ell \rightarrow R$. These were also derived earlier by Wilkinson,³⁵ Prasad,³⁶ and Wilkinson and Kalnins.³⁷

9.1 FIRST EQUATION

$$L_{uu}\bar{u} = \sqrt{B} \frac{\partial^2}{\partial \eta^2} [\sqrt{B} \bar{u}] + F(2 - \kappa)\bar{u} + \frac{F}{B} \frac{\partial^2 \bar{u}}{\partial \phi^2}, \quad (9.1)$$

$$L_{uv}\bar{v} = (1 + F) \frac{\eta}{B} \frac{\partial \bar{v}}{\partial \phi} + (v + F) \frac{\partial^2 \bar{v}}{\partial \eta \partial \phi}, \quad (9.2)$$

$$L_{uw}\bar{w} = (1 + v + \kappa F) \sqrt{B} \frac{\partial \bar{w}}{\partial \eta}, \quad (9.3)$$

$$L_{u\beta_\eta} \beta_\eta = F\kappa \beta_\eta, \quad (9.4)$$

$$L_{u\beta_\phi} \beta_\phi = 0,$$

and

$$\text{Right side} = \frac{R^2}{c_p^2} \left\{ (1 + \varepsilon) \ddot{\bar{u}} + 2\varepsilon \ddot{\beta}_\eta \right\} - \frac{1 - v^2}{Eh} R q_u. \quad (9.5)$$

9.2 SECOND EQUATION

$$L_{vu}\bar{u} = (v + F) \frac{\partial^2 \bar{u}}{\partial \eta \partial \phi} - (1 + F) \frac{\eta}{B} \frac{\partial \bar{u}}{\partial \phi}, \quad (9.6)$$

$$L_{vv}\bar{v} = F\sqrt{B} \frac{\partial^2}{\partial \eta^2} [\sqrt{B} \bar{v}] + F(2 - \kappa)\bar{v} + \frac{1}{B} \frac{\partial^2 \bar{v}}{\partial \phi^2}, \quad (9.7)$$

$$L_{vw}\bar{w} = (1 + v + \kappa F) \frac{1}{\sqrt{B}} \frac{\partial \bar{w}}{\partial \phi}, \quad (9.8)$$

$$L_{v\beta_\eta} \beta_\eta = 0,$$

$$L_v \beta_\phi \beta_\phi = F \kappa \beta_\phi, \quad (9.9)$$

$$\text{Right side} = \frac{R^2}{c_p^2} \left\{ (1 + \varepsilon) \ddot{v} + 2\varepsilon \ddot{\beta}_\phi \right\} - \frac{1 - v^2}{Eh} R q_v. \quad (9.10)$$

9.3 THIRD EQUATION

$$L_{wu} \bar{u} = -(1 + v + \kappa F) \frac{\partial}{\partial \eta} [\sqrt{B} \bar{u}], \quad (9.11)$$

$$L_{wv} \bar{v} = -(1 + v + \kappa F) \frac{1}{\sqrt{B}} \frac{\partial \bar{v}}{\partial \phi}, \quad (9.12)$$

$$L_{ww} \bar{w} = F \kappa \frac{\partial}{\partial \eta} \left[B \frac{\partial \bar{w}}{\partial \eta} \right] + \frac{F \kappa}{B} \frac{\partial^2 \bar{w}}{\partial \phi^2} - 2(1 + v) \bar{w}, \quad (9.13)$$

$$L_{w\beta_\eta} \beta_\eta = F \kappa \frac{\partial}{\partial \eta} [\sqrt{B} \beta_\eta], \quad (9.14)$$

$$L_{w\beta_\phi} \beta_\phi = F \kappa \frac{1}{\sqrt{B}} \frac{\partial \beta_\phi}{\partial \phi}, \quad (9.15)$$

$$\text{Right side} = \frac{R^2}{c_p^2} (1 + \varepsilon) \ddot{w} - \frac{1 - v^2}{Eh} R q_w. \quad (9.16)$$

9.4 FOURTH EQUATION

$$L_{\beta_\eta u} \bar{u} = F \kappa \bar{u}, \quad (9.17)$$

$$L_{\beta_\eta v} \bar{v} = 0,$$

$$L_{\beta_\eta w} \bar{w} = -F \kappa \sqrt{B} \frac{\partial \bar{w}}{\partial \eta}, \quad (9.18)$$

$$L_{\beta_\eta \beta_\eta} \beta_\eta = -F \kappa \beta_\eta + \varepsilon \left\{ \frac{\partial}{\partial \eta} \left[B \frac{\partial \beta_\eta}{\partial \eta} \right] + F \frac{1}{B} \frac{\partial^2 \beta_\eta}{\partial \phi^2} - \left[v + \frac{\eta^2}{B} \right] \beta_\eta \right\}, \quad (9.19)$$

$$L_{\beta_\eta \beta_\phi} \beta_\phi = \varepsilon \left\{ (v + F) \frac{\partial^2 \beta_\phi}{\partial \eta \partial \phi} + (1 + F) \frac{\eta}{B} \frac{\partial \beta_\phi}{\partial \phi} \right\}, \quad (9.20)$$

$$\text{Right side} = \frac{R^2}{c_p^2} \varepsilon \left\{ \ddot{\beta}_\eta + 2\ddot{u} \right\} - \frac{1 - v^2}{Eh} m_{\beta_\eta}. \quad (9.21)$$

9.5 FIFTH EQUATION

$$L_{\beta_\phi u} \bar{u} = 0,$$

$$L_{\beta_\phi v} \bar{v} = F\kappa \bar{v}, \quad (9.22)$$

$$L_{\beta_\phi w} \bar{w} = -F\kappa \frac{1}{\sqrt{B}} \frac{\partial \bar{w}}{\partial \phi}, \quad (9.23)$$

$$L_{\beta_\phi \beta_\eta} \beta_\eta = \varepsilon \left\{ (v + F) \frac{\partial^2 \beta_\eta}{\partial \eta \partial \phi} - (1 + F) \frac{\eta}{B} \frac{\partial \beta_\eta}{\partial \phi} \right\}, \quad (9.24)$$

$$L_{\beta_\phi \beta_\phi} \beta_\phi = -F\kappa \beta_\phi + \varepsilon \left\{ F \frac{\partial}{\partial \eta} \left[B \frac{\partial \beta_\phi}{\partial \eta} \right] + \frac{1}{B} \frac{\partial^2 \beta_\phi}{\partial \phi^2} + F \left[1 - \frac{\eta^2}{B} \right] \beta_\phi \right\}, \quad (9.25)$$

$$\text{Right side} = \frac{R^2}{c_p^2} \varepsilon \left\{ \ddot{\beta}_\phi + 2\ddot{v} \right\} - \frac{1 - v^2}{Eh} m_{\beta_\phi}. \quad (9.26)$$

10. SUMMARY

This report has documented the derivation of the equations of motion of PSS of constant thickness. The equations were derived from the Lagrangian of the system, including the strain and kinetic energies and the potential for external mechanical surface forces and moments. The strain energy density was derived for three independent displacements and two changes-of-curvature. The strain energy was developed in a Taylor series of thickness-to-radius ratios of up to third order. Thus, the theory includes the effects of membrane, bending, and thickness-shear deformations. The kinetic energy density includes the translational as well as rotational components, so that the equations of motion include translational and rotatory inertias. The potential for external forces includes distributed surface forces and moments.

Use of Hamilton's principle on the Lagrangian of the system resulted in five coupled partial differential equations for nonaxisymmetric vibration. These equations are self-adjoint and positive definite due to the use of the Lagrangian and the calculus of variations in their derivation. The five partial differential equations are coupled for non-axisymmetric motion but reduce to two systems for axisymmetric vibration. One system has three coupled partial differential equations on u , w , and β_η , i.e., nontorsional displacement fields and the second system has two coupled partial differential equations on v and β_ϕ , i.e., torsional displacement fields.

The thin-shell theory used in this report results in a frequency-wavenumber spectrum with five branches. The accuracy of the thin shell theory is judged on how accurate these branches agree with the exact elasticity theory. It has been known that the inclusion of the thickness-shear and rotatory inertia effects makes the lowest flexural branch have the correct group and phase velocities in the high-frequency/high-wavenumber limit. The second acoustic branch represents torsional motion and it also has the correct phase and group velocities in the high-frequency limit. The first optical branch represents longitudinal motion, where the group and phase velocities are approximately correct. The remaining two optical branches represent the thickness-shear modes.

These equations were reduced to those for the nonaxisymmetric vibration of spherical shells and shown to agree with those developed earlier by various authors.

The five coupled partial differential equations have nonconstant coefficients and do not have closed form eigenfunctions that would satisfy the system of equations. In future reports, approximate comparison functions will be used for the five dependent variables u , v , w , β_η , and β_ϕ . These would result in coupled $5N \times 5N$ algebraic equations in terms of N comparison function sets.

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APPENDIX
COMPENDIUM OF USEFUL FORMULAE FOR
PROLATE SPHEROIDAL SHELLS

In this appendix, the following variables will be used:

$$A = a^2 - 1, \quad B = 1 - \eta^2, \quad C = a^2 - \eta^2, \quad D = \frac{\sqrt{a^2 - 1}}{\sqrt{a^2 - \eta^2}}, \quad \text{and} \quad G = 2/d.$$

Thus,

$$h_\eta h_\phi = \left(\frac{d}{2}\right)^2 \sqrt{AC},$$

$$\frac{h_\phi}{h_\eta} = B \sqrt{\frac{A}{C}},$$

$$\frac{\partial h_\phi}{\partial \eta} = -\frac{d}{2} \eta \sqrt{\frac{A}{C}},$$

$$\frac{1}{h_\eta h_\phi} \frac{\partial h_\phi}{\partial \eta} = -G \frac{\eta}{\sqrt{BC}},$$

$$\frac{\partial}{\partial \eta} \left(\frac{1}{h_\eta h_\phi} \right) = G^2 \frac{\eta}{C^{3/2} \sqrt{A}},$$

$$\frac{1}{h_\eta} \frac{\partial h_\phi}{\partial \eta} = -\eta D,$$

$$\frac{1}{h_\phi} \frac{\partial h_\phi}{\partial \eta} = -\frac{\eta}{B},$$

$$\frac{\partial}{\partial \eta} \left(\frac{h_\phi}{h_\eta} \right) = -\eta D(1 + D^2),$$

$$\frac{\partial}{\partial \eta} \left(\frac{1}{h_\eta} \frac{\partial h_\phi}{\partial \eta} \right) = -a^2 \frac{D}{C},$$

$$r_{\eta} = \frac{1}{R_{\eta}} = aG \frac{D}{C},$$

$$r_{\phi} = \frac{1}{R_{\phi}} = aG \frac{1}{\sqrt{AC}},$$

$$r_{\eta} r_{\phi} = G^2 \frac{a^2}{C^2},$$

$$r_{\eta} - r_{\phi} = -aG \frac{B}{\sqrt{AC}^{3/2}},$$

$$r_{\eta} + r_{\phi} = aG \frac{1}{\sqrt{AC}} \left(2 - \frac{B}{C} \right),$$

$$\frac{\partial r_{\eta}}{\partial \eta} = 3aG \eta \frac{D}{C^2},$$

$$\frac{\partial r_{\phi}}{\partial \eta} = \frac{aG \eta}{\sqrt{A} C^{3/2}},$$

$$\frac{r_{\eta}}{h_{\eta}} = G^2 \frac{a \sqrt{AB}}{C^2},$$

$$\frac{r_{\phi}}{h_{\phi}} = G^2 \frac{a}{A \sqrt{BC}},$$

$$\frac{r_{\phi}}{h_{\eta}} = G^2 \frac{a \sqrt{B}}{C \sqrt{A}},$$

$$(r_{\eta} - r_{\phi}) h_{\eta} h_{\phi} = -a \frac{d}{2} \frac{B}{C},$$

$$r_{\eta} (r_{\eta} - r_{\phi}) = -a^2 G^2 \frac{B}{C^3},$$

$$r_{\phi}(r_{\eta}-r_{\phi})=-a^2G^2\frac{B}{AC^2},$$

$$r_{\eta}(r_{\eta}-r_{\phi})h_{\eta}h_{\phi}=-a^2\frac{BD}{C^2},$$

$$r_{\phi}(r_{\eta}-r_{\phi})h_{\eta}h_{\phi}=-a^2\frac{B}{\sqrt{A}C^{3/2}},$$

$$(r_{\eta}-r_{\phi})\frac{h_{\phi}}{h_{\eta}}=-aG\frac{B^2}{C^2},$$

$$\frac{\partial}{\partial\eta}\left[(r_{\eta}-r_{\phi})\frac{h_{\phi}}{h_{\eta}}\right]=aG\frac{4\eta AB}{C^3},$$

$$\frac{\partial}{\partial\eta}A^{-n/2}=n\eta A^{-(n-2)/2}, \quad n=1,3,5,\dots,$$

$$\frac{\partial}{\partial\eta}A^{n/2}=-n\eta A^{(n-2)/2}, \quad n=1,3,5,\dots,$$

$$\frac{C}{A}=1+\frac{B}{A}, \quad \frac{B}{A}=-1+\frac{C}{A},$$

$$\frac{A}{C}=1-\frac{B}{C}, \quad \frac{B}{C}=1-\frac{A}{C},$$

$$\frac{\eta^2}{C}=\frac{a^2}{C}-1,$$

$$\frac{A}{BC}=\frac{1}{B}-\frac{1}{C},$$

$$\frac{\eta^2}{C}=\frac{1}{A}\left(1-a^2\frac{B}{C}\right).$$

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